# Harmonic Functions Overview

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Harmonic functions are an important area of study not only in mathematics, but also in physics, engineering, and chemistry among many other fields. However, the definition of harmonic functions is typically restricted to Euclidean space, forcing one to use coordinates when describing these functions. In his paper, The Geometry of Harmonic Functions, Tristan Needham describes formulating the solution to a harmonic function using only geometry. His work makes uses of the Mean Value Property of harmonic functions: given a harmonic function, a point, and a disk centered at the point, the value of the point can be determined by the average of the circumference. However, his work is restricted to R2, so we wish to extend this geometric idea of harmonic functions to Rn. To do this, we will employ the use of Mobius space and similarity geometry.

Mobius geometry is defined by the Mobius transformation. This transformation is defined on the extended complex plane by the function f(z) = (az + b)/(cz + d), where z is a complex variable, a, b, c, d are complex constants, and ad-bc is not equal to 0. Then, the set of all Mobius transformations form a group and the Mobius geometry is the pair (M, C), where C denotes the extended complex plane. Geometrically, Mobius geometry describes unoriented spheres in Rn. Mobius space is equivalent to the Mobius sphere which is defined by stereographic projections onto the plane.

Similarity geometry is typically described as Euclidean geometry with a scaling transformation in addition to the Euclidean transformation group. From Felix Klein's view point, an n-dimensional similarity space is given by an n-dimensional real projective space with a distinguished hyperplane, an n-1 dimensional plane that lives in n-dimensional space, and an (n-2)-dimensional imaginary quadric C embedded in the hyperplane. Projective transformations that leave these absolutes invariant precisely give the Eulidean group with scaling transformation. Klein's characterization is a top-down approach. The projective space is the most general geometry, and the absolutes eliminate the transformation group down to the similarity group.

Additionally, we have a concept called the Kelvin Transform, which acts on spheres in Rn. Using the Kelvin Transform and the mean value property of harmonic functions we have shown that we can generate the Poisson formula, which can recover values of a harmonic function inside a sphere based solely on the value of the harmonic function on the boundary. This is important because this allows us to define harmonic functions on the Mobius sphere.

We will then define distinct points, p, q, on the Mobius sphere, and using stereographic projection, project these points onto different families of similarity geometries. Through this, we should be able to define a way to correspond these distinct points, and this way should be in correspondence to the Poisson formula and the Kelvin transform in some way.

Overall, harmonic functions are useful for many fields of study, but especially in robotics where they are used to generate smooth paths. Thus, having an alternative method to define harmonic functions may open new pathways in that field. Ultimately, the main purpose of this research project is to add to the theory of harmonic functions which is a large area of study in mathematics called potential theory.