Analyst's Traveling Salesman Problem

Fencil, Blase; Jeong, Minyoung; Koirala, Robert; Li, Mica.

June 2025

1 Introduction

In the traditional, combinatorial formulation of the Traveling Salesman Problem, we are asking the following question: if we have a list of cities and the distances between each pair of cities, what is the shortest route possible where each city is visited exactly once and we return to the city we originated from? This problem lies more in the realm of theoretical computer science and combinatorial optimization and is a popular and well-studied problem in those fields.

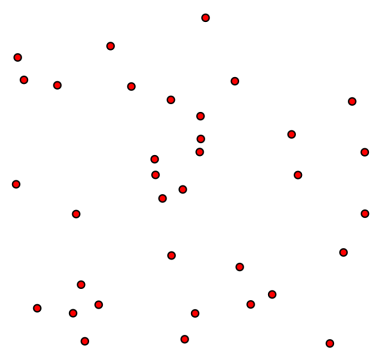

Figure 1: Unsolved traveling salesman problem

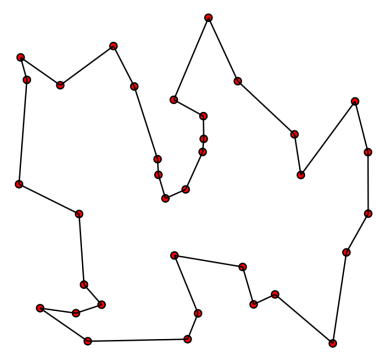

Figure 2: Solution to the traveling salesman problem

In this survey paper, we will be focusing on a generalization of the Traveling Salesman Problem, called the analyst's Traveling Salesman Problem.

This version is similar to the original problem, but it is over an arbitrary set of cities, meaning that we can also ask questions like: given a set K, what is the shortest curve that contains it?

This question and others that came from it were first discussed by Peter Jones, where the analyst's version was proved in the Euclidean plane, with the use of Jones $\beta-$ numbers. These results were then improved by Okikiolu [4] where the results were proved in arbitrary $\mathbb{R}^n$. Other results were also given by Raanan Schul, who proved the theorem to be true in Hilbert spaces [2] and

Immo Hahlomaa, who extended the proof to general metric spaces using Menger curvature [6].

Jones proved the case of $\mathbb{R}^2$ [3] using techniques mainly from complex analysis. In generalizing to $\mathbb{R}^n$, and other spaces, Okikiolu, Schul, and Hahlomaa all took more geometric approaches. Before discussing these approaches and proofs, we first need to make some definitions.

2 Definitions

Let $\mathcal{M}$ denote a metric space with distance function $\text{dist} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$. We will be concerned with sets $K \subset M$ such that $\text{diam}(K) < \infty$.

Definition 1 (ϵ-nets and Multi-resolution Family)**.** *A set $X \subset K$ is an ϵ-net for K if*

(i) $\forall\, x_1, x_2 \in X$ *we have* $\text{dist}(x_1, x_2) > \epsilon$

(ii) $\forall\, y \in K,\ \exists\, x \in X$ *such that* $\text{dist}(x, y) \le \epsilon$

For a set X let X_n^K denote a sequence of 2^{-n} -nets such that $X_n^K \subset X_{n+1}^K$. We define the multi-resolution family as:

$$D^k := \{\text{Ball}(x, A2^{-n}) \mid x \in X_n^K,\ n \in \mathbb{Z},\ n\ >\ n_0\}.$$

For constants $A > A_0 > 1$ and integer n_0.

Definition 2 (Rectifiable Sets and Curves)**.** *A set $S \subset \mathbb{R}^n$ is k-rectifiable if:*

$$\mathcal{H}^k(S \backslash \bigcup_i f_i(S_i)) = 0.$$

Here $f_i : S_i \subseteq \mathbb{R}^k \to \mathbb{R}^n$ is a countable collection of Lipschitz functions. A rectifiable curve is a set that is the image of compact interval under a Lipschitz function.

3 Analyst's Traveling Salesman Problem in $\mathbb{R}^n$

3.1 Jones β-Numbers

In order to state the existence and optimality results for the analyst's traveling salesman problem, let us define the Jones β-Numbers:

Definition 3 (Jones β-Numbers)**.** *For $K \subset \mathbb{R}^n$, we consider the set Q that is a ball or a cube. We define $\beta_{\infty,K}$ as:*

$$\beta_{\infty,K}(Q) = \frac{1}{\text{diam}(Q)} \inf_{L\ line} \sup_{x \in K \cap Q} d(x, L).$$

Given a locally finite measure μ and $1 < p < \infty$, we additionally define:

$$\beta_p(Q;\mu) := \frac{1}{\operatorname{diam}(Q)} \inf_{L \text{ line}} \left(\int_Q (d(x,L))^p \frac{d\mu(y)}{\mu(Q)} \right)^{1/p}.$$

The Jones β-number is a scale-invariant quantity which measures non-flatness of K or support of μ.

3.2 Existence and optimality

We state the existence of (nearly) optimal solution to the Analyst's traveling salesman problem.

Theorem 1 (Existence)**.** *Suppose A_0 is large enough. Given a set $K \subset \mathbb{R}^n$, there exists a connected set $\Gamma_0 \subset K$ such that the length of Γ_0 satisfies:*

$$\mathcal{H}^1(\Gamma_0) \lesssim \operatorname{diam}(K) + \sum_{\mathcal{D}^K} \beta^2_{\infty,\Gamma}(Q) \operatorname{diam}(Q)$$

Theorem 2 (Optimality)**.** *For any connected set $\Gamma \subset \mathbb{R}^n$ such that $\Gamma \supseteq K$ we have:*

$$\sum_{\mathcal{D}^K} \beta^2_{\infty,\Gamma}(Q) \operatorname{diam}(Q) \lesssim \mathcal{H}^1(\Gamma)$$

It is well known that for any connected $\Gamma \subseteq \mathbb{R}^n$, there exists a path $\gamma : I \to \Gamma$ which is surjective and the length of γ is $2\,\mathcal{H}^1(\Gamma)$. Therefore, the length of a solution to the analyst's traveling salesman problem is comparable to

$$\min\{\mathcal{H}^1(\Gamma) : \Gamma \text{ is connected}, K \subseteq \Gamma\}$$

By the two results stated above, this is comparable to

$$\operatorname{diam}(K) + \sum_{\mathcal{D}^K} \beta^2_{\infty,\Gamma}(Q) \operatorname{diam}(Q)$$

3.3 Sketch of the proof

Proof sketch of existence. The goal is to construct a connected set $\Gamma_0 \supset K$ whose 1-dimensional Hausdorff measure is controlled by the sum of squared Jones β-numbers across a multiresolution family of cubes. The approach builds on the geometric idea that a rectifiable curve is, in some averaged sense, flat at most locations and scales.

We begin by covering the compact set $K \subset \mathbb{R}^n$ using a multiresolution family $\mathcal{D}^K$ of dyadic cubes, constructed via a nested sequence of ε-nets. For each cube $Q \in \mathcal{D}^K$, the β_∞-number measures how far the set $\Gamma \cap Q$ deviates from being contained in a line. The total deviation over all cubes, weighted by $\operatorname{diam}(Q)$, serves as a proxy for the "complexity" of connecting K.

To construct the curve Γ_0, we analyze K at multiple scales using a multiresolution family of dyadic cubes. In those cubes where the β-number is small, the intersection $\Gamma \cap Q$ is close to a line segment, and we can connect points using short paths that follow a best-fitting line. The sum $\sum_Q \beta^2(Q)\,\mathrm{diam}(Q)$ captures the total contribution from these nearly flat regions. In the remaining "non-flat" cubes, where the set deviates more significantly from a straight line, the construction is more delicate: geometric decompositions, dyadic packing, and straight-segment approximations are used to join points while still keeping the total added length under control. The key is to show that even these irregular regions do not contribute excessively to the total length, allowing us to bound $\mathcal{H}^1(\Gamma_0)$ by a multiple of $\mathrm{diam}(K) + \sum_Q \beta^2(Q)\,\mathrm{diam}(Q)$.

Finally, the local segments are pieced together into a connected set Γ_0 contained in K. A covering argument ensures that the construction does not create excessive overlap, and the resulting curve satisfies

$$\mathcal{H}^1(\Gamma_0) \lesssim \mathrm{diam}(K) + \sum_{Q \in \mathcal{D}^K} \beta^2_{\infty,\Gamma}(Q)\,\mathrm{diam}(Q),$$

as desired.

□

Proof sketch of optimality. From results in abstract graphs and a compactness property, we get a Lipschitz parametrization of Γ, $\gamma : [0,1] \to \Gamma$ where the Lipschitz norm is bounded by a constant times the length – given by one dimensional Hausdorff measure – of Γ. This γ is then fixed throughout the proof.

We begin by first assuming that for every $x \in \Gamma$ and for $r \geq 0$, $\gamma^{-1}(\mathrm{Ball}(x,r))$ has only one connected component. By assuming this we can actually change the question from one of bounding $\sum \beta^2(Q)\mathrm{diam}(Q)$, to one of the geometry of the image under γ of a multiresolution on the domain of γ. Then using successive approximations and inequality in the Euclidean case or using the definition of β in the metric case, one is able to get the inequality.

The case where the above assumption cannot be made is far more difficult to deal with. With such cases, one can consider the case where $\Gamma \cap Q$ is a collection of straight line segments where the endpoints are outside the ball. With this consideration, the proof can proceed. It was here the Jones used complex analysis techniques and Okikiolu used geometric ones. Schul was able to extend these techniques even further, and we will be using their language. For a ball Q that satisfies these conditions, a weight w_Q can be assigned to the set $\Gamma \cap Q$. This weight is assigned in such a way that for every Q,

$$\int w_Q d\mathcal{H}^1|_\Gamma \geq \beta^2(Q)\mathrm{diam}(Q) \qquad (1)$$

and for every x, $\sum_Q w_Q(x) \leq 1$. In doing so, $\int w_Q d\mathcal{H}^2|_\Gamma$ controls $\beta(Q)\mathrm{diam}(Q)$. The sum and integral can then be exchanged, giving us the desired bound. □

3.4 Ahlfors regularity

By making an additional regularity assumption on K, we can strengthen the previous results. To begin with, we define

Definition 4 (Ahlfors-Regularity)**.** *A set K is k-Ahlfors-Regular if there is a $C > 0$ such that for all $x \in K$ and $0 < r < \operatorname{diam}(K)$ we have:*

$$\frac{r^k}{C} \leq \mathcal{H}^k|_K(B_r(x)) \leq Cr^k.$$

Using Ahlfors-Regularity, David-Semmes [5] proved the following result which is a variation of theorems 1 and 2.

Theorem 3 Let $K \subset \mathbb{R}^d$ be a 1-Ahlfors-Regular set and $1 \leq q \leq \infty$. Then K is contained in a connected 1-Ahlfors-Regular set if and only if for all $z \in K$ and $0 < R < \operatorname{diam}(K)$

$$\int_0^R \int_{\mathrm{Ball}(z,R)} \beta_{q,\mathcal{H}^1|_K}(\mathrm{Ball}(x,t))^2 \, d\mathcal{H}^1|_K(x) \, \frac{dt}{t} \lesssim R.$$

Proof Sketch. First, suppose that K is contained in a connected 1-Ahlfors-Regular set Γ_0. In this case, most small-scale views of Γ_0 resemble straight line segments, and the local flatness β_q is small outside a controlled collection of scales and positions. Using a packing argument, we find that the collection of balls where flatness is large satisfies a Carleson condition, leading to a bounded square function integral. Since $K \subset \Gamma_0$, the same integral bound applies when the integration is restricted to K.

Conversely, suppose the square function estimate holds for K. The goal is to construct a connected 1-Ahlfors-Regular curve Γ_0 containing K. The proof uses a stopping-time argument and corona decomposition: at each scale, balls (or cubes) are classified as "good" when the local flatness is small and "bad" otherwise. In good regions, the set is nearly flat and can be locally connected using short segments approximating the geometry of K. The bad regions are controlled by the Carleson-type bound implicit in the square function estimate, which ensures that their contribution to the total length is quantitatively small.

By systematically connecting nearby points using these local structures and patching the components through both good and bad regions, one constructs a connected set Γ_0 containing K. The construction ensures that Γ_0 is 1-Ahlfors-Regular.

□

4 Generalizations

These ideas were extended even further by Hahlomaa who was able to show affirmative results in a general metric spaces using something called Menger curvature.

Definition 5 (Menger Curvature)**.** *Let $x_1, x_2, x_3 \in \mathbb{R}^m$ be non-collinear points, and let R be the radius of the circle that contains these points. We define the*

Menger curvature to be the reciprocal of the this radius:

$$c(x_1, x_2, x_3) = \frac{1}{R}.$$

If x_1, x_2, x_3 are collinear then we define $c(x_1, x_2, x_3) = 0$.

Theorem 4 Suppose A_0 is large enough. Let $\mathcal{M}$ be a metric space. Let $K \subset \mathcal{M}$. Let $Q \in D^K$. Define

$$\beta^2_{\mathcal{M},\infty,K}(Q)\operatorname{diam}(Q) = \operatorname{diam}(Q)^3 \sup_{\substack{x_1,x_2,x_3 \in Q \\ \operatorname{dist}(x_i,x_j) \geq A^{-1}\operatorname{diam}(Q)}} c^2(x_1, x_2, x_3).$$

Then there exists $K' \subset [0,1]$ and a function $f : K' \to K$ such that

$$\|f\|_{Lip} \lesssim \operatorname{diam}(K) + \sum_{D^K} \beta^2_{\mathcal{M},\infty,K}(Q)\operatorname{diam}(Q)$$

and $\operatorname{Image}(f) = K$.

Proof sketch. The main idea is to construct a sequence of approximating graphs G_j with controlled total length and edge weights derived from Menger curvature. These graphs are built iteratively using a net $(\Delta_k)_{k\in\mathbb{Z}}$ of K, and each G_j connects a finite number of points with special care to preserve ordering and control angles (to respect the geometric curvature condition). The graphs are updated using four combinatorial cases depending on local geometric configurations and the behavior of the Menger curvature function β.

Each step ensures that the total graph length does not increase too quickly and that a connected structure is maintained. A key technical step is showing that the constructed graphs G_j satisfy certain geometric and combinatorial properties, allowing the control of the Lipschitz constant in the limit.

Eventually, the graphs converge in a suitable sense, and a limit Lipschitz map f is defined from $[0,1]$ to K. The Lipschitz constant of this map is bounded by a constant multiple of $\beta(K) + \operatorname{diam}(K)$, completing the proof.

□

5 Conclusion

The Analyst's TSP is a generalization of the classical TSP, extending it from a finite, discrete setting to arbitrary metric spaces. Through the use of techniques and concepts such as the Jones' $\beta-$number, Ahlfors-Regularity, and Menger curvature, we can characterize and control the rectifiability of sets.

These ideas, using the Jones' $\beta-$number was first introduced by Peter Jones, and then furthered by Okikiolu, Schul, Hahlomaa, and many others.

References

[1] Raanan Schul. *Analyst's Traveling Salesman Theorems. A Survey.*

[2] Raanan Schul. *Subsets of rectifiable curves in Hilbert space–the analyst's TSP*. Journal d'Analyse Mathématique, **103**:331–375, 2007.

[3] Peter W. Jones. *Rectifiable sets and the Traveling Salesman Problem.* Inventiones Mathematicae, 102:1–15, 1990.

[4] Kate Okikiolu. Characterization of subsets of rectifiable curves in $\mathbb{R}^n$. *J. London Math. Soc. (2)*, **46**(2):336–348, 1992.

[5] Guy David and Stephen Semmes. *Singular integrals and rectifiable sets in* $\mathbb{R}^n$. *Au-delà des graphes lipschitziens.* Astérisque, tome 193, 1991.

[6] Immo Hahlomaa. Menger curvature and Lipschitz parametrizations in metric spaces. *Fund. Math.*, pages 143–169, 2005.