Closest Point Exterior Calculus

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1 INTRODUCTION

Solving Partial Differential Equations (PDEs) on surfaces is a crucial component in geometry processing with many applications ranging from texture generation to calculating distance on a curved surface. Among the many methods to solving PDEs on surfaces, a method which uses a triangular mesh along with exterior calculus called Discrete Exterior Calculus (DEC) [Hirani 2003] (and similarly Finite Element Exterior Calculus [Arnold 2018]) that is particularly elegant due to its coordinate-free nature and inherent differential operators. However, a limitation of DEC – and any mesh-based PDE solver – is that the method relies on a decent mesh quality.

From our experiments (see Fig. 1), when the triangulation of a mesh is poor, the solution to the surface PDE differs significantly from the reference solution computed on a higher quality mesh. Directly tackling the problem of poor mesh quality (*i.e.* extrinsic or intrinsic remeshing) is non-trivial [Sharp et al. 2021].

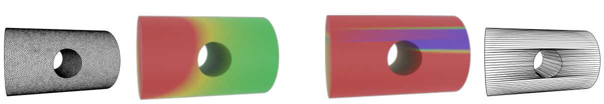

Figure 1: Heat flow simulated with DEC on a high quality mesh (left) and on a poor mesh (right). Wireframe renders of the mesh are shown next to each simulation.

There exists another method, the Closest Point Method (CPM), proposed by Ruuth and Merriman [2008] and recently brought into attention in computer graphics [King et al. 2023], that is able to bypass the need of a mesh to solve PDEs on surfaces. In this method, one solves a *reformulation* of the PDE on the neighborhood of the surface instead of directly on the surface. A closest point map is used for extending surface functions to the neighborhood. The main benefit of employing this implicit approach is that a regular Cartesian grid can be used, enabling high order methods without needing to worry about mesh quality. However, this closest point reformulation has been explored for a limited class of differential expressions involving scalar functions and occasionally vector fields. Fundamental objects in coordinate-free calculus such as differential forms and their exterior calculus operations have not been studied in the closest point framework.

In this work, we propose a novel method that unifies the strengths of DEC with the CPM, called Closest Point Exterior Calculus (CP-EC). We provide a general framework for reformulating a surface exterior calculus expression to its equivalent expression in the 3D neighborhood using the closest point map. Furthermore, the supported exterior calculus operators include the wedge and the interior products, which are operators that have been challenging to discretize on a triangle mesh. Our CP-EC apparatus allows one to robustly represent a more complete set of operators in exterior calculus while also having the flexibility to work with curved surface independent of the mesh quality.

2 BACKGROUND

Exterior calculus is the calculus for differential forms. The fundamental operators are the wedge product ($\wedge$), the interior product ($i_{\vec{v}}$) with respect to a vector $\vec{v}$, the exterior derivative d, and the Hodge star ($\star$) [Wang et al. 2023]. While these operators have abstract definitions based on axiomatic rules on general manifolds, in a 3D Cartesian space, they become familiar vector calculus operators. Table 1 summarizes this correspondence using the subscript notation: $f_{(k)}$ converts a 1D or 3D array of numbers f into a k-form. Table 2 expresses the exterior calculus operators in terms of vector calculus using this correspondence.

One main advantage of exterior calculus over vector calculus is that most of the operators commute with *pullbacks*. A smooth map $\varphi\colon M \to N$ between two manifolds M and N induces a canonical *pullback* $\varphi^*\colon \Omega^k(N) \xrightarrow{\text{linear}} \Omega^k(M)$ on the spaces of k-forms that represents the change of variables for integral evaluations

$$\int_{\varphi(S)} \alpha =: \int_S \varphi^*\alpha, \quad \forall\, \alpha \in \Omega^k(N), k\text{-dim surface } S \subset M. \tag{1}$$

The pullback operator distributes over wedge products $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$, commutes with interior products $\varphi^* i_{d\varphi(\vec{u})}\alpha = i_{\vec{u}}\varphi^*\alpha$, and commutes with the exterior derivative $\varphi^*(d\alpha) = d(\varphi^*\alpha)$. Note that we do not have similar identities for vector calculus operators $(\cdot, \times, \nabla)$ under general changes of variables.

3 THEORY

We develop the theory for closest point exterior calculus by exploiting the commutativity properties between pullbacks and exterior

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Symbol	Definition	Meaning
$f_{(0)}$	f	0-form (scalar field)
$\mathbf{u}_{(1)}[\![\mathbf{v}_{\mathrm{vec}}]\!]$	$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$	1-form (field of linear functions on vectors)
$\mathbf{u}_{(2)}[\![\mathbf{v}_{\mathrm{vec}}, \mathbf{w}_{\mathrm{vec}}]\!]$	$\det\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$	2-form (skew-symmetric bilinear form of vectors)
$f_{(3)}[\![\mathbf{u}_{\mathrm{vec}}, \mathbf{v}_{\mathrm{vec}}, \mathbf{w}_{\mathrm{vec}}]\!]$	$f\det\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$	3-form (scalar times the volume form)

Table 1: Differential forms in the 3D Cartesian space.

Output type	Wedge product ($\wedge$)	Interior product ($i_{\vec{u}}$), ($\vec{u} := \mathbf{u}_{\mathrm{vec}}$)	Exterior derivative (d)	Hodge star ($\star$)
0-form	$f_{(0)} \wedge g_{(0)} = (fg)_{(0)}$	$i_{\vec{u}}\mathbf{w}_{(1)} = (\mathbf{u} \cdot \mathbf{w})_{(0)}$	N/A	$\star f_{(3)} = f_{(0)}$
1-form	$f_{(0)} \wedge \mathbf{u}_{(1)} = f\mathbf{u}_{(1)}$	$i_{\vec{u}}\mathbf{w}_{(2)} = (\mathbf{w} \times \mathbf{u})_{(1)}$	$df_{(0)} = (\nabla f)_{(1)}$	$\star\mathbf{u}_{(2)} = \mathbf{u}_{(1)}$
2-form	$\mathbf{u}_{(1)} \wedge \mathbf{v}_{(1)} = (\mathbf{u} \times \mathbf{v})_{(2)}$	$i_{\vec{u}}f_{(3)} = f\mathbf{u}_{(2)}$	$d\mathbf{u}_{(1)} = (\nabla \times \mathbf{u})_{(2)}$	$\star\mathbf{u}_{(1)} = \mathbf{u}_{(2)}$
3-form	$\mathbf{u}_{(1)} \wedge \mathbf{v}_{(2)} = (\mathbf{u} \cdot \mathbf{v})_{(3)}$	N/A	$d\mathbf{u}_{(2)} = (\nabla \cdot \mathbf{u})_{(3)}$	$\star f_{(0)} = f_{(3)}$

Table 2: Exterior calculus operators in 3D.

calculus operators. The theory essentially characterizes an isomorphism between the calculus on a surface and the calculus in a 3D neighborhood.

Let S be a surface embedded in $\mathbb{R}^3$ by the inclusion map $j\colon S \hookrightarrow N \subset \mathbb{R}^3$, where $N \subset \mathbb{R}^3$ is a neighborhood of $j(S)$. The *closest point function* $cp\colon N \to S$ takes a point in this neighborhood and returns the closest point on the surface. Conversely, the pullback operator $j^*\colon \Omega^k(N) \xrightarrow{\text{linear}} \Omega^k(S)$ extracts the tangential part of a 3D k-form field at the surface, while $cp^*\colon \Omega^k(S) \xrightarrow{\text{linear}} \Omega^k(N)$ extends a surface k-form to the neighborhood N. Note that for a scalar function $f_{(0)}$ on S, $cp^* f_{(0)} = f \circ cp$ is the constant-along-normal extension in the classical CPM. The endomorphism $(j \circ cp)^* = cp^* j^*\colon \Omega^k(N) \xrightarrow{\text{linear}} \Omega^k(N)$ replaces a neighborhood k-form by its extension from its value at the surface $j(S)$.

In practice, N is represented by a regular grid. The pullback j^* is an *interpolator*, allowing us to query $(j^*\alpha)(p)$ at an arbitrary off-grid surface point p using the grid data $\alpha \in \Omega^k(N)$. We use the cubic Lagrange interpolation as in the classical CPM. The pullback cp^* of the closest point map in $(j \circ cp)^* = cp^* j^*$ amounts to a post-multiplication by a 3×3 matrix related to the Jacobian $\mathbf{F} = d(j \circ cp)$:

$$cp^* j^* f_{(0)} = (f \circ cp \circ j)_{(0)} \tag{2a}$$

$$cp^* j^* \mathbf{u}_{(1)} = (\mathbf{F}^\intercal \mathbf{u} \circ cp \circ j)_{(1)} \tag{2b}$$

$$cp^* j^* \mathbf{u}_{(2)} = (\mathrm{cof}(\mathbf{F})^\intercal \mathbf{u} \circ cp \circ j)_{(2)} \tag{2c}$$

$$cp^* j^* f_{(3)} = (\det(\mathbf{F}) f \circ cp \circ j)_{(3)} = 0. \tag{2d}$$

Here, $\mathrm{cof}(\mathbf{F})$ denotes the cofactor matrix of $\mathbf{F}$, and $\det(\mathbf{F}) = 0$ since cp maps from 3D to 2D. Note that each of the composition evaluator $(\cdot) \circ cp \circ j$ is identical to the extension operator for 0-forms in the classical CPM. The generalization of $cp^* j^*$ to k-forms is significant, since many of the operators in exterior calculus work well with each other through this pullback. In particular, cp^* allows us to emulate the operators on S (denoted with a superscript S) using the operators on $\mathbb{R}^3$ (denoted with a superscript $\mathbb{R}^3$) which we already have exact arithmetic expressions (Table 2):

$$\text{CP-wedge product:} \quad cp^*(\alpha \wedge^S \beta) = (cp^*\alpha) \wedge^{\mathbb{R}^3} (cp^*\beta), \tag{3a}$$

$$\text{CP-interior product:} \quad cp^*(i^S_{\mathbf{Fu}}\alpha) = i^{\mathbb{R}^3}_{\mathbf{u}} cp^*\alpha, \tag{3b}$$

$$\text{CP-exterior derivative:} \quad cp^*(d^S\alpha) = d^{\mathbb{R}^3} cp^*\alpha, \tag{3c}$$

$$\text{CP-Hodge star:} \quad cp^*(\star^S\alpha)\big|_{j(S)} = i_{\vec{n}} \star^{\mathbb{R}^3} (cp^*\alpha)\big|_{j(S)}. \tag{3d}$$

Note that the CP-Hodge star is only applicable directly at the surface (due to the metric dependency of $\star$) while other CP-EC operators work on the surface *and* in the neighborhood of the surface.

These new operators (3) acting on the subspace $\mathrm{im}(cp^*) = \mathrm{im}(cp^* j^*) \subset \Omega^\bullet(N)$ of differential form spaces, assisted with the subspace projection $cp^* j^*$ given by (2), allows one to translate differential equations and expressions on surfaces to a CPM framework.

4 DISCUSSION

In the past, each CPM requires individual analysis for converting a surface PDE to the 3D neighborhood. With this work we formulate an effective language which rapidly translates multivariable calculus PDE problems formulated on surfaces into closest point problems. This is made possible by formalizing the CPM using the concept of pullbacks and utilizing the commutativity of pullbacks with exterior calculus operators. The CP-EC also improves exterior-calculus-based methods as the wedge product and interior product were difficult to discretize on a triangle mesh, and other exterior calculus operators relied on the quality of the mesh.

We hope that this poster work will inspire exciting computational methods for surface PDEs, as implicit representations become popular in recent years. We also hope that our unifying framework allows more concise numerical analysis in the future: Convergence properties can be analyzed for each fundamental operators, instead for each individual PDE problems independently. Ultimately, CP-EC forms a language that categorially maps a surface PDE problem into the closest point world, allowing a deeper understanding of the correspondence between an implicit and an explicit representations.

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